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Sign k-potent sign patterns and ray k-potent ray patterns that allow k-potence

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ABSTRACT

In this paper, we characterize sign k-potent sign pattern matrices that allow k-potence. In particular, the structure of a sign idempotent sign pattern matrix that allows idempotence is given. Thus an open problem posed by Eschenbach is affirmatively solved. We also extend these results to ray k-potent ray pattern matrices, providing the structure of a ray k-potent ray pattern matrix that allows k-potence.

1. Introduction

A matrix whose entries consist of $+, -$ and 0 is called a sign pattern matrix. Let $S^{n\times n}$ be the set of all $n \times n$ sign pattern matrices. For the sake of statement convenience, write $A = +$ if all entries of A are $+$. For $A = (a_{ij}) \in S^{n\times n}, A^m$ is defined as a sign pattern if for all i and j, no two nonzero terms in the sum

$$(A^m)_{ij} = \sum_{t_1,\ldots,t_{m-1}} a_{i,t_1} \cdots a_{t_{m-1},j} \tag{1}$$

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are oppositely signed, i.e., all entries of A^m are unambiguous. The square sign pattern matrix A is powerful if A^m is unambiguous for every positive integer m. If $A = A^{k+1}$ with the positive integer k minimal, then the square sign pattern matrix A is called sign k-potent. In particular, if $k = 1$, then A is called sign idempotent. The following result in [11] reveals a connection between sign k-potent sign pattern matrices and powerful sign pattern matrices.

Lemma 1 [11]. *Let A be an irreducible sign pattern matrix. If A is sign k-potent, then A is powerful.*

A generalized permutation matrix is either a permutation matrix or a matrix obtained by replacing some or all of the 1 entries in a permutation matrix with nonzero entries. Let P denote a generalized permutation matrix each of whose nonzero entries is 1 or -1. Obviously, $P^{-1} = P^T$. Let $R^{n\times n}$ be the set of all $n \times n$ real matrices. If $B = B^{k+1}$ with the positive integer k minimal, then the square real matrix B is said to be k-potent. For a sign pattern matrix $A \in S^{n\times n}$, define

$$Q(A) = \{B \in R^{n\times n} | \mathrm{sign}(B) = A\}.$$

If there exists a k-potent matrix $B \in Q(A)$, then the sign k-potent sign pattern matrix A is said to allow k-potence. We also say that the sign k-potent sign pattern matrix A is realized by a k-potent matrix B. However, not all sign k-potent sign pattern matrices allow k-potence. For example, the sign idempotent sign pattern matrix

$$A_1 = \begin{pmatrix} + & - \\ 0 & + \end{pmatrix}$$

does not allow idempotence. Thus, identifying sign idempotent sign pattern matrices that allow idempotence is an open problem posed by Eschenbach in [2]. In [4] we considered the problem and obtained the result as follows. *Let A be a sign idempotent sign pattern matrix with no zero diagonal entries. Then A allows idempotence if and only if there exists a generalized permutation matrix P such that*

$$P^TAP = \begin{pmatrix} A_{11} & & \\ & \ddots & \\ & & A_{rr} \end{pmatrix},$$

where each $A_{ii} = +$ is square.

Motivated by the above problem, we study sign k-potent sign pattern matrices which allow k-potence. We require the important result from [11] as follows.

Lemma 2 [11]. *Let A be an irreducible sign k-potent sign pattern matrix. Then there exists a generalized permutation matrix P such that*

$$P^TAP = \begin{pmatrix} 0 & J_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & J_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & J_{m-1} \\ \gamma J_m & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

where the diagonal blocks are square and each $J_i = +$. Moreover, $m = k$ or $\frac{k}{2}$, and $\gamma = \begin{cases} + & \text{if } m = k, \\ - & \text{if } m = \frac{k}{2}. \end{cases}$

(Note that when $m = 1, P^TAP = \pm J_1$.)

Next we will give the structures of sign k-potent sign pattern matrices that allow k-potence. In particular, the open problem posed by Eschenbach is affirmatively solved as our corollary. Finally, some generalizations of these results are provided.

2. Sign k-potent sign pattern matrices

By Lemma 2, it is easily checked that the following result holds.

Theorem 3. *Let $A \in S^{n\times n}$ be an irreducible sign k-potent sign pattern matrix. Then A allows k-potence, i.e., there exists a k-potent matrix $B \in Q(A)$ which is generalized permutation similar to a matrix of the form*

$$\begin{pmatrix} 0 & \alpha_1\beta_1^T & 0 & \cdots & 0 & 0 \\ 0 & 0 & \alpha_2\beta_2^T & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \alpha_{m-1}\beta_{m-1}^T \\ \gamma\alpha_m\beta_m^T & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{2}$$

where all diagonal blocks are square, and all α_i and β_i are positive column vectors for which the product $(\beta_1^T\alpha_2)\cdots(\beta_m^T\alpha_1) = 1$. Moreover, $m = k$ or $\frac{k}{2}$, and $\gamma = \begin{cases} 1 & \text{if } m = k, \\ -1 & \text{if } m = \frac{k}{2}. \end{cases}$ (Note that when $m = 1$, B is generalized permutation similar to $\pm\alpha_1\beta_1^T$ with $\beta_1^T\alpha_1 = 1$ for positive column vectors α_1 and β_1.)

As in the proof of Lemma 1 in [10], we easily get the result as follows.

Lemma 4. *Let A be a sign k-potent sign pattern matrix. Then $A = A^{tk+1}$ for all positive integers t. In particular, A^{tk+1} is unambiguous for all positive integers t. Further, if $A = A^{p+1}$ for some positive integer p, then k divides p.*

Group the row indices of A into four sets according to whether the ith row and column are both nonzero, the ith row is zero and ith column is nonzero, the ith row is nonzero and the ith column is zero, or both the ith row and column are zero. Then A is permutation similar to the following matrix:

$$\begin{pmatrix} C & D & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E & F & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3}$$

where all diagonal blocks are square, C and D have no common zero rows, and C and E have no common zero columns. Note that $A = A^{k+1}$ is equivalent to the four statements:

$$C = C^{k+1}, \quad D = C^kD, \quad E = EC^k, \quad F = EC^{k-1}D.$$

Lemma 5. *Let $A \in S^{n\times n}$ be a sign k-potent sign pattern matrix in the form (3). Then C is a sign k-potent sign pattern matrix with no zero rows and no zero columns.*

Proof. Note that the fact that A is sign k-potent implies that $C = C^{k+1}$. Now assume that C is sign t-potent. Next we prove that $t = k$.

If $t = 1 < k$, then $C = C^2$, consequently, $C^{k+1} = C^{t+1}$, $C^k = C$, and $C^{k-1} = C$. Thus $A = A^2$, so by minimality, we get a contradiction. If $1 < t < k$, by Lemma 4, then $k = tr$ for some positive integer r, and

$$C^{k+1} = C^{rt+1} = C^{t+1}, C^k = C^{(r-1)t+1+(t-1)} = C^t,$$

and since $t \geq 2$, $C^{k-1} = C^{(r-1)t+1+(t-2)} = C^{t-1}$. Thus $A = A^{t+1}$, so by minimality, we get a contradiction. Hence $t = k$. This means that C is sign k-potent.

Since $D = C^kD$, by the fact that C and D have no common zero rows, C has no zero rows. Similarly, since $E = EC^k$, C has no zero columns. $\square$

Lemma 6. *Let $B \in R^{n\times n}$ be a nonnegative matrix with no zero rows and no zero columns. If $B = B^{k+1}$, then there exists a permutation matrix P such that*

$$P^TBP = \mathrm{diag}(B_1, \ldots, B_d)$$

where each B_i is irreducible and square.

Proof. If B is irreducible, obviously the conclusion holds. Let B be reducible. To get the result, we use induction on the order n of B. The case $n = 1$ is trivial. Now assume the result is true for matrices of order less than n. Since B is reducible, without loss of generality, let

$$B = \begin{pmatrix} B_1 & B_{12} \\ 0 & B_2 \end{pmatrix},$$

where B_1 and B_2 are square. Since $B = B^{k+1}$, we have that

$$B_{12} = B_1^k B_{12} + B_1^{k-1} B_{12} B_2 + \cdots + B_1 B_{12} B_2^{k-1} + B_{12} B_2^k,$$

and $B_i = B_i^{k+1}$ for $i = 1, 2$. Hence,

$$B_1 B_{12} B_2 = B_1 B_{12} B_2 + \cdots + B_1 B_{12} B_2,$$

which implies that $B_1 B_{12} B_2 = 0$ for B to be nonnegative. Thus, by the fact that B_1 has no zero columns and B_2 has no zero rows, we have $B_{12} = 0$. Therefore, applying the induction assumption to B_1 and B_2, we get that the result holds. $\square$

Lemma 7 [8]. *Let $A = \mathrm{diag}(A_1, \ldots, A_d)$ where each A_i is sign k_i-potent. Then A is sign k-potent if and only if $k = \mathrm{lcm}(k_1, \ldots, k_d)$.*

Lemma 8 [5]. *Let $B = \mathrm{diag}(B_1, \ldots, B_r)$ where each B_i is k_i-potent. Then B is k-potent if and only if $k = \mathrm{lcm}(k_1, \ldots, k_r)$.*

It is important to notice that $|B| = |B|^{k+1}$ if a sign k-potent sign pattern matrix A is realized by a k-potent matrix $B = (b_{ij}) \in Q(A)$ according to the definition (1), where $|B| = (|b_{ij}|)$.

Theorem 9. *Let $A \in S^{n\times n}$ be sign k-potent. Then A allows k-potence if and only if there exists a generalized permutation matrix P such that*

$$P^TAP = \begin{pmatrix} M & Y & 0 & 0 \\ 0 & 0 & 0 & 0 \\ X & XM^{k-1}Y & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4}$$

where all diagonal blocks are square and M is a sign k-potent sign pattern matrix with $M = \mathrm{diag}(M_1, M_2, \ldots, M_d)$ for some positive integer d, where each M_i is an irreducible sign k_i-potent sign pattern matrix with $k = \mathrm{lcm}(k_1, \ldots, k_d)$, and

$$M_i = \begin{pmatrix} 0 & J_1^{(i)} & 0 & \cdots & 0 & 0 \\ 0 & 0 & J_2^{(i)} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & J_{m_i-1}^{(i)} \\ \gamma_i J_{m_i}^{(i)} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{5}$$

where each diagonal block, denoted by $\widetilde{M}_{jj}^{(i)}$, is $n_{jj}^{(i)} \times n_{jj}^{(i)}$ and each $J_j^{(i)} = +$ for $j = 1, 2, \ldots, m_i$. Moreover, $m_i = k_i$ or $\frac{k_i}{2}$, and $\gamma_i = \begin{cases} + & \text{if } m_i = k_i, \\ - & \text{if } m_i = \frac{k_i}{2}. \end{cases}$ (Note that if $m_i = 1$, then $M_i = \pm J_1^{(i)}$.) In addition, $Y = (Y_j^{(i)})$ is a row partitioned block matrix where the row index set of $Y_j^{(i)}$ is the same as that of $\widetilde{M}_{jj}^{(i)}$, and each column of $Y_j^{(i)}$ is positive, negative or zero. Similarly, $X = (X_j^{(i)})$ is a column partitioned block matrix where the column *index set of $X_j^{(i)}$ is the same as that of $\widetilde{M}_{jj}^{(i)}$, and each row of $X_j^{(i)}$ is positive, negative or zero, and $XM^{k-1}Y$ is unambiguous.*

Proof. First assume that A allows k-potence. Then there exists a k-potent matrix $B=(b_{ij})\in Q(A)$. Note that generalized permutation similarity preserves sign k-potence. Without loss of generality, we assume that

$$A=\begin{pmatrix} M & Y & 0 & 0\\ 0 & 0 & 0 & 0\\ X & F & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \tag{6}$$

where all diagonal blocks are square, and M is a sign k-potent sign pattern matrix with no zero rows and no zero columns by Lemma 5. Obviously, B has the same form (6) as A. By the definition (1), no two nonzero terms in the sum

$$(B^{k+1})_{ij}=\sum_{t_1,\ldots,t_k} b_{i,t_1}b_{t_1,t_2}\cdots b_{t_k,j}$$

are oppositely signed for all i and j, which means that $|B|=|B|^{k+1}$. By Lemma 6 applied to $|B|$, it is not difficult to check that we can assume that

$$M=\mathrm{diag}(M_1,\ldots,M_d),$$

where each M_i is an irreducible sign k_i-potent sign pattern matrix, and $k=\mathrm{lcm}(k_1,\ldots,k_d)$ by Lemma 7. By Lemma 2, we assume that M_i is in the form (5). According to the form (5), we have that $M^k=\mathrm{diag}(M_1^k,\ldots,M_d^k)$ where

$$M_i^k=\begin{pmatrix} M_{11}^{(i)} & & & \\ & M_{22}^{(i)} & & \\ & & \ddots & \\ & & & M_{m_i,m_i}^{(i)}\end{pmatrix}$$

with each $M_{jj}^{(i)}=+$ is $n_{jj}^{(i)}\times n_{jj}^{(i)}$ for $j=1,\ldots,m_i$. Let $Y=(Y_j^{(i)})$ be a row partitioned block sign pattern matrix according to M, where the row index set of $Y_j^{(i)}$ is the same as that of $M_{jj}^{(i)}$. Since $M^kY=Y$, we have $M_{jj}^{(i)}Y_j^{(i)}=Y_j^{(i)}$ which implies that each column of $Y_j^{(i)}$ is positive, negative or zero. Similarly, let $X=(X_j^{(i)})$ be a column partitioned block sign pattern matrix according to M, where the column index set of $X_j^{(i)}$ is the same as that of $M_{jj}^{(i)}$. Since $XM^k=X$, we have $X_j^{(i)}M_{jj}^{(i)}=X_j^{(i)}$ which implies that each row of $X_j^{(i)}$ is positive, negative or zero. Since A is k-potent, $F=XM^{k-1}Y$ is unambiguous. Thus the conclusion holds.

Conversely, assume that the sign k-potent sign pattern matrices A is generalized permutation similar to the form (4) and (5). We will prove that there does exist a k-potent matrix $B\in Q(A)$. Since M is a sign k-potent sign pattern matrix with $M=\mathrm{diag}(M_1,M_2,\ldots,M_d)$ where each M_j is of the form given by (5), there exists $T=\mathrm{diag}(T_1,T_2,\ldots,T_d)\in Q(M)$, where each $T_i\in Q(M_i)$ is an irreducible k_i-potent matrix in the form given by (2) from Theorem 3. Since $k=\mathrm{lcm}(k_1,\ldots,k_d)$, T is k-potent by Lemma 8. Thus we assume that

$$P^TBP=\begin{pmatrix} T & Y_1 & 0 & 0\\ 0 & 0 & 0 & 0\\ X_1 & F_1 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

Next we assert that there exists a real matrix $Y_1\in Q(Y)$ such that $T^kY_1=Y_1$. By the form (2), we have that $T^k=\mathrm{diag}(T_1^k,\ldots,T_d^k)$ where

$$T_i^k = \begin{pmatrix} T_{11}^{(i)} & & & \\ & T_{22}^{(i)} & & \\ & & \ddots & \\ & & & T_{m_i,m_i}^{(i)} \end{pmatrix}$$

with each $T_{jj}^{(i)} > 0$ is $n_{jj}^{(i)} \times n_{jj}^{(i)}$. The fact that $T = T^{k+1}$ implies that the spectral radius $\rho(T_{jj}^{(i)}) = 1$ for $j = 1, \ldots, m_i$. Hence, by Perron–Frobenius Theorem [1], there exists a positive or negative vector y such that $T_{jj}^{(i)} y = y$, from which it is easily obtained that there does exist a real matrix $Y_1 \in Q(Y)$ such that $T^k Y_1 = Y_1$. Similarly, we get that there does exist a real matrix $X_1 \in Q(X)$ such that $X_1 T^k = X_1$. Let $F_1 = X_1 T^{k-1} Y_1$. Then $B \in Q(A)$ is k-potent. This means that A allows k-potence. $\square$

Corollary 10. *Let A be a sign idempotent sign pattern matrix. Then A allows idempotence if and only if there exists a generalized permutation matrix P such that*

$$P^T A P = \begin{pmatrix} J_1 & & & & Y_1 & 0 & 0 \\ & J_2 & & & Y_2 & 0 & 0 \\ & & \ddots & & \vdots & \vdots & \vdots \\ & & & J_d & Y_d & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ X_1 & X_2 & \cdots & X_d & \sum_{i=1}^{d} X_i Y_i & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{7}$$

where all diagonal blocks are square and each $J_i = +$. Moreover, every column of each Y_i is positive, negative or zero; every row of each X_i is positive, negative or zero; and $\sum_{i=1}^{d} X_i Y_i$ is unambiguous.

Proof. By Theorem 9, note that $k = 1$ since A is sign idempotent, then A is generalized permutation similar to the form (7). $\square$

3. Extensions to ray k-potent ray pattern matrices

A ray pattern matrix is a matrix each of whose entries is either 0 or a ray in the complex plane of the form $re^{i\theta}$, where $\theta \in R$ and r runs through all positive real numbers. Ray pattern matrices are natural generalizations of sign pattern matrices. For brevity, we denote a ray $re^{i\theta}$ simply by $e^{i\theta}$. Of course, $e^{i\theta} = e^{i(\theta+2k\pi)}$ for any integer k; if the arguments of two rays do not differ by an integer multiple of 2π, then the rays are distinct. The product of two rays is given by $e^{i\theta_1} e^{i\theta_2} = e^{i\theta_1 + i\theta_2}$. For the addition, if θ_1 and θ_2 differ by a multiple of 2π, then $e^{i\theta_1} + e^{i\theta_2} = e^{i\theta_1}$; otherwise any sum of two or more distinct rays results an ambiguous argument. For more results and notations, the reader is referred to [3,6,7,9].

Let $C^{n\times m}$ be the set of all $n \times m$ complex matrices. For an $n \times m$ ray pattern matrix $A = (a_{ij})$, define

$$\mathscr{N}(A) = \{B \in C^{n\times m} | b_{ij} = 0 \text{ iff } a_{ij} = 0;\ \arg(b_{ij}) = \arg(a_{ij}) \text{ otherwise}\}.$$

A ray pattern matrix A is called ray k-potent if $A = A^{k+1}$ with the positive integer k minimal. If there exists a k-potent complex matrix $B \in \mathscr{N}(A)$, then the ray k-potent ray pattern matrix A is said to allow k-potence. Next we will characterize ray k-potent ray pattern matrices that allow k-potence, which is a natural generalization of these results about sign k-potent sign pattern matrices.

Let Q be a generalized permutation matrix each of whose nonzero entries is a complex number of unit modulus. We require the following result from [10] with a slight modification, which generalizes Lemma 2.

Lemma 11 [10]. *Let A be an irreducible ray k-potent ray pattern matrix. Then there exists a generalized permutation matrix Q such that*

$$Q^TAQ = \begin{pmatrix} 0 & W_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & W_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & W_{m-1} \\ \eta W_m & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

where the diagonal blocks are square and each W_i is an all ones matrix. Moreover, m divides k, and η is a primitive (k/m)th root of unity. (Note that when $m = 1, Q^TAQ = \eta W_1$.)

By Lemma 11, it is easily checked that the following result holds.

Theorem 12. *Let A be an irreducible ray k-potent ray pattern matrix. Then A allows k-potence, i.e., there exists a k-potent matrix $B \in \mathcal{N}(A)$ which is generalized permutation similar to a matrix of the form*

$$\begin{pmatrix} 0 & \alpha_1\beta_1^T & 0 & \cdots & 0 & 0 \\ 0 & 0 & \alpha_2\beta_2^T & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \alpha_{m-1}\beta_{m-1}^T \\ \eta\alpha_m\beta_m^T & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

where all diagonal blocks are square, and all α_i and β_i are positive column vectors for which the product $(\beta_1^T\alpha_2)\cdots(\beta_m^T\alpha_1) = 1$. Moreover, m divides k, and η is a primitive (k/m)th root of unity.(Note that when $m = 1$, B is generalized permutation similar to $\eta\alpha_1\beta_1^T$ with $\beta_1^T\alpha_1 = 1$ for positive column vectors α_1 and β_1.)

Lemma 13 [10]. *Let A be a ray k-potent ray pattern matrix. Then $A = A^{tk+1}$ for all positive integers t. In particular, A^{tk+1} is unambiguous for all positive integers t. Further, if $A = A^{p+1}$ for some positive integer p, then k divides p.*

The following lemmas are easily obtained analogously to the proofs of Lemmas 5 and 7, respectively.

Lemma 14. *Let A be an $n \times n$ ray k-potent ray pattern matrix in the form (3). Then C is a ray k-potent ray pattern matrix with no zero rows and no zero columns.*

Lemma 15. *Let $A = \mathrm{diag}(A_1, \ldots, A_d)$ where each A_i is a ray k_i-potent ray pattern matrix. Then A is ray k-potent if and only if $k = \mathrm{lcm}(k_1, \ldots, k_d)$.*

Note that $|B| = |B|^{k+1}$ if a ray k-potent ray pattern matrix A is realized by a k-potent matrix $B = (b_{ij}) \in \mathcal{N}(A)$ according to the definition of the addition and multiplication of two or more rays, where $|B| = (|b_{ij}|)$. Therefore, these results about sign k-potent sign pattern matrices naturally extend to ray k-potent ray pattern matrices as follows by a similar argument to that of Theorem 9.

Theorem 16. *Let A be an $n \times n$ ray k-potent ray pattern matrix. Then A allows k-potence if and only if there exists a generalized permutation matrix Q such that*

$$Q^TAQ = \begin{pmatrix} M & Y & 0 & 0 \\ 0 & 0 & 0 & 0 \\ X & XM^{k-1}Y & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

where all diagonal blocks are square and M is a ray k-potent ray pattern matrix with $M = \mathrm{diag}(M_1, M_2, \ldots, M_d)$ for some positive integer d, where each M_i is an irreducible ray k_i-potent ray pattern matrix with $k = \mathrm{lcm}(k_1, \ldots, k_d)$, and

$$M_i = \begin{pmatrix} 0 & W_1^{(i)} & 0 & \cdots & 0 & 0 \\ 0 & 0 & W_2^{(i)} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & W_{m_i-1}^{(i)} \\ \eta_i W_{m_i}^{(i)} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

where each diagonal block, denoted by $\widetilde{M}_{jj}^{(i)}$, *is* $n_{jj}^{(i)} \times n_{jj}^{(i)}$ *and each* $W_j^{(i)}$ *is an all ones matrix for* $j = 1, 2, \ldots, m_i$. *Moreover,* m_i *divides* k_i, *and* η_i *is a primitive* (k_i/m_i)*th root of unity. (Note that if* $m_i = 1$, *then* $M_i = \eta_i W_1^{(i)}$.*) In addition,* $Y = (Y_j^{(i)})$ *is a row partitioned block matrix where the row index set of* $Y_j^{(i)}$ *is the same as that of* $\widetilde{M}_{jj}^{(i)}$, *and each column of* $Y_j^{(i)}$ *is a multiple of the all ones vector. Similarly,* $X = (X_j^{(i)})$ *is a column partitioned block matrix where the column index set of* $X_j^{(i)}$ *is the same as that of* $\widetilde{M}_{jj}^{(i)}$, *and each row of* $X_j^{(i)}$ *is a multiple of the all ones vector, and* $XM^{k-1}Y$ *is unambiguous.*

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